Vector Equations

We have a compact/abstract way to write/study linear systems. This involves vectors.

For now, vector means an ordered list of numbers.

Vectors in $\mathbb{R}^2$

A matrix of one column is a (column) vector.

Exmp vectors w/ two entries

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

v_1 and v_2 are real #'s.

$\mathbb{R}^2$: the set of all vectors w/ two entries.

$\mathbb{R}$: the set of real #'s.

Equality : Two vectors are equal if their corresponding entries are equal.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Sum $\vec{u} + \vec{v}$, add corresponding entries

e.g. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Scaler multiplication

$c \in \mathbb{R}$ (scalar) $\vec{u} \in \mathbb{R}^2$

$c\vec{u}$: multiply each entry of $\vec{u}$ by c.

e.g. $c = 2$, $\vec{u} = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$

$$c\vec{u} = \begin{bmatrix} 2 \times 0.5 \\ 2 \times 0.4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

Exmp (addition & scalar mult.)

(Kept on the board)

Properties of addition and scalar multiplication

addition:
- (associative) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
- (commutative) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.
- (zero vector) $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{u}+\vec{0} = \vec{u}$
- (inverse) $\vec{u}+(-\vec{u}) = \vec{0}$ — we can define $(-\vec{u})$

scalar mult.:
- $(ab)\vec{u} = a(b\vec{u})$
- $1\vec{u} = \vec{u}$

between addition and scalar mult.:
- $(a+b)\vec{u} = a\vec{u}+b\vec{u}$
- $a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$.

Exmp. $\vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ find $5\vec{u}$, $9\vec{v}$, $5\vec{u}+9\vec{v}$.

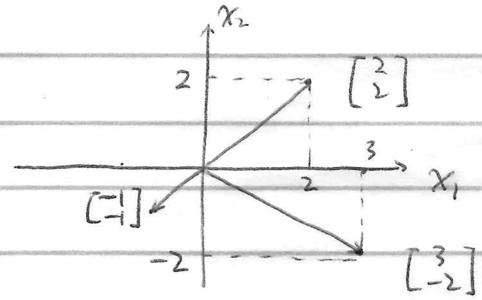
Sol $5\vec{u} = \begin{bmatrix} 5\times 1 \\ 5\times(-3) \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \end{bmatrix}$, $9\vec{v} = \begin{bmatrix} 9\times 3 \\ 9\times 7 \end{bmatrix} = \begin{bmatrix} 27 \\ 63 \end{bmatrix}$

$$5\vec{u}+9\vec{v} = \begin{bmatrix} 5 \\ -15 \end{bmatrix} + \begin{bmatrix} 27 \\ 63 \end{bmatrix} = \begin{bmatrix} 5+27 \\ -15+63 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \end{bmatrix}$$ $\square$

Geometric descriptions of $\mathbb{R}^2$

Consider a rectangular coordinate system in the plane.
Each column vector in $\mathbb{R}^2$ corresponds to a point in the plane.

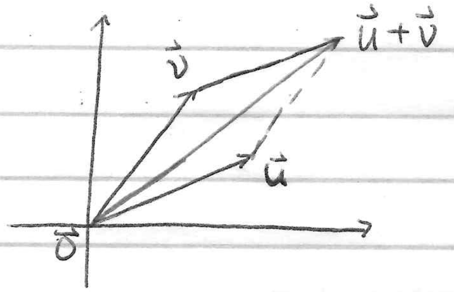
Exmp.



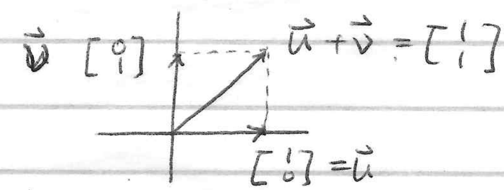
Quite often, we denote them as arrows

Parallelogram rule for addition

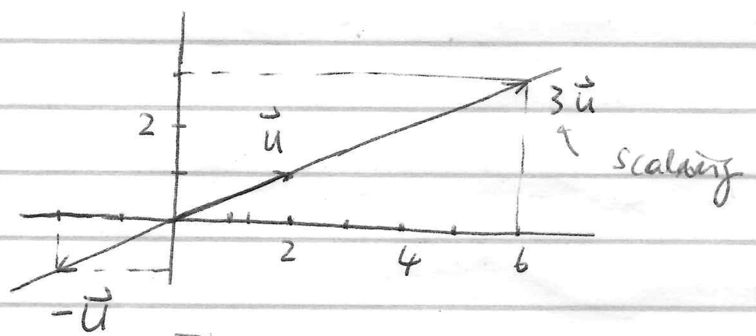
If $\vec{u}$ and $\vec{v}$ are represented as points in the plane, then $\vec{u}+\vec{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\vec{0}$, $\vec{u}$, $\vec{v}$.



Exmp $\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$



Exmp $\vec{u}=\begin{bmatrix}2\\1\end{bmatrix}$, display $\vec{u}, 3\vec{u}$, and $-\vec{u}$



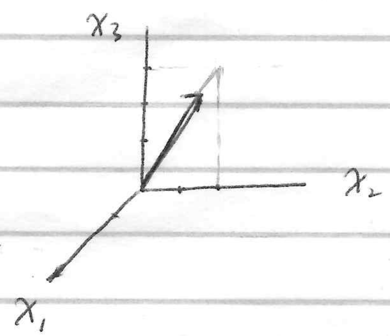
The minus sign reverses the direction of the arrow.

Vectors in $\mathbb{R}^3$

These are 3×1 matrices.

Geometric representation:

e.g. $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in $\mathbb{R}^3$



Vectors in $\mathbb{R}^n$

The set of all lists of ~~ordered~~ n real numbers. We write them as $n\times 1$ matrices such as $\vec{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$

Zero vector: $\vec{0}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

Addition and scalar multiplication are similar to those in $\mathbb{R}^2$.

Properties have been listed before.

$\boxed{\text{Notation}}$ $-\vec{u}$ means $(-1)\vec{u}$

$\vec{u} + (-\vec{v}) = \vec{u} - \vec{v}$

Linear Combinations

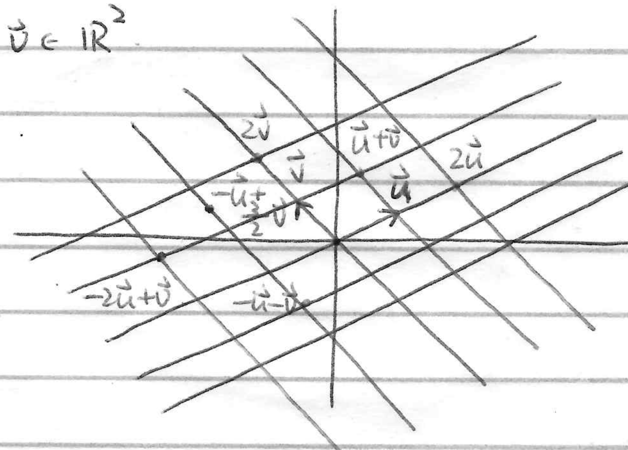
Given vectors $\vec{v}_1, \cdots, \vec{v}_p \in \mathbb{R}^n$ (belongs to) and scalars $c_1, \cdots, c_p (\in \mathbb{R})$

$$\vec{y} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$$

is the linear combination of $\vec{v}_1, \cdots, \vec{v}_p$ w/ weights $c_1, \cdots, c_p$.

Because addition is associative, we don't need to put parentheses in the expression above. We don't need to be concerned about the order of additions

Exmp $\vec{u}, \vec{v} \in \mathbb{R}^2$.



Exmp. $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Determine whether b can be written as a linear combination of $\vec{a}_1$ and $\vec{a}_2$. That is, are there $x_1, x_2 \in \mathbb{R}$ s.t.

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b} \qquad (*)$$

If yes, find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Sol Equation $(*)$ is the same as $x_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

the same as $\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

the same as $\begin{bmatrix} x_1+2x_2 \\ -2x_1+5x_2 \\ -5x_1+6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

It is equivalent the ~~system~~ system

$$\begin{cases} x_1+2x_2=7 \\ -2x_1+5x_2=4 \\ -5x_1+6x_2=-3 \end{cases}$$

We solve the linear system by row reducing the argmented matrix

$$\begin{array}{ccc} & & \\ \left[\begin{matrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{matrix}\right] \\ \uparrow \quad \uparrow \quad \uparrow \\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{b} \end{array} = [\vec{a}_1, \vec{a}_2, \vec{b}] \quad \text{for brevity.}$$

The Solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Thus. $\vec{b} = 3\vec{a}_1 + 2\vec{a}_2$. ☐

Linear Systems v.s. vector equations

A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has the same solution set as the linear system whose augmented matrix is $[\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n, \vec{b}]$

In particular, $\vec{b}$ can be written as a linear combination of $\vec{a}_1, \cdots, \vec{a}_n$ if and only if the linear system has a solution

The following is important.

Defn $\vec{v}_1, \cdots, \vec{v}_p \in \mathbb{R}^n$. The subset of $\mathbb{R}^n$ spanned by $\vec{v}_1, \cdots, \vec{v}_p$ is

$$\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\} = \{c_1\vec{v}_1 + \cdots + c_p\vec{v}_p \mid \underbrace{c_1, \cdots, c_p \in \mathbb{R}}_{\text{scalars}}\}$$

(equivalent to

$\vec{b}$ belongs to $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ $\Leftrightarrow$

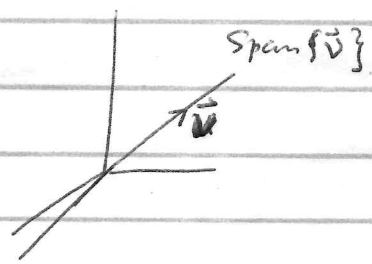
vector equation $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{b}$ has a solution $\Leftrightarrow$

linear system w/ augmented matrix $[\vec{v}_1 \dots \vec{v}_p \ \vec{b}]$ has a sol.

<u>Note</u> $\vec{0} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$

<u>Geometric description of Span</u>

<u>Exmp</u> $\vec{v} \in \mathbb{R}^3$, $\neq \vec{0}$. $\text{Span}\{\vec{v}\}$ = line through $\vec{0}$ and $\vec{v}$.



<u>Exmp</u> $\vec{u}, \vec{v} \in \mathbb{R}^3$ nonzero and $\vec{v}$ is not a multiple of $\vec{u}$.

$\text{Span}\{\vec{u}, \vec{v}\}$ = plane containing $\vec{0}$, $\vec{u}$ and $\vec{v}$.

In particular, it contains the line through $\vec{0}$ and $\vec{u}$,

the line through $\vec{0}$ and $\vec{v}$.