Plan

- (Reduced) echlon forms
- Pivot positions & columns
- Row reduction algorithm.
- Solutions of linear systems
- Existence & uniqueness

Row reduction & echlon forms

We refine the method in the previous section into a row reduction algorithm.

nonzero row / column : a row/ column containing at least one nonzero entry.

leading entry of a row : leftmost nonzero entry (in a nonzero row)

Show it on the screen

Def A matrix is in **echlon form** (or row echlon form) if

1. All nonzero rows are above any zero rows
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

$2 \Rightarrow 3$

We want nontrivial equations to be on the top.

We want to eliminate x_i from the equations below.

A matrix is in reduced echlon form, if in addition

4. The leading entry in each nonzero row is 1
5. Each leading 1 is the only nonzero entry in its column.

Examp.

$$\begin{bmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & 0 & 0 & \boxed{\frac{5}{2}} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

echlon matrix — reduced echlon matrix

We may transform a matrix by applying elementary row operations into different echelon matrices. But

Thm (Uniqueness of the reduced echelon form)

Each matrix is row equivalent to one and only one reduced echelon matrix.

In particular, we have pivot positions.

Pivot positions

Since the reduced echelon form is unique.

the leading entries are always in the same positions in any echelon form obtained from a given matrix.

Def. These positions are pivot positions.

(Important.) The corresponding columns are pivot columns.

Exmp Locate the pivot positions & columns of

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

KEEP ON THE BOARD.

Solution Row reduction

$$A \xrightarrow{\text{row } 1 \leftrightarrow \text{row } 4} \begin{array}{c} \text{pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{pivot column} \end{array}$$

The top of the leftmost nonzero column is the first pivot position

operate on row 2 & 3.

$$\xrightarrow[\text{row } 3 + 2 \times \text{row } 1]{\text{row } 2 + \text{row } 1} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot.

pivot column

clear out entries below the first pivot

Look at the 2nd row, we find the 2nd pivot.

pivot column. pivot

$$\xrightarrow[\text{row } 4 + \frac{3}{2} \times \text{row } 2]{\text{row } 3 - \frac{5}{2} \times \text{row } 2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

← all 0's

Warning The pivots in the example are 1, 2, and −5. They are different from the entries of A in those positions.

In summary

The row reduction algorithm

Screen

forward phase

Step 1 Begin w/ leftmost nonzero column. This is a pivot column. The pivot position is at the top

Step 2 Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 3 Use row replacement operations to create zeros in all positions below the pivot.

Step 4 Cover the row containing the pivot positions and cover all rows, if any, above it. Apply steps 1−3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

(backward phase)

Step 5 Beginning w/ the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, (row) scale it to 1.

This reduces the # of cal.

What is left to be done in the exmp.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row } 3 \div (-5)} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{row } 1 + 9 \times \text{row } 3]{\text{row } 2 + 6 \times \text{row } 3} \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row } 2 \div 2}$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row } 1 - 4 \times \text{row } 2} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction Algorithm $\Longrightarrow$

Solutions of Linear Systems

For exmp. suppose the augmented matrix has been transforms into the equivalent reduced echlon form.

pivot columns

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -5 & 1 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3 variables.

assoc. linear sys. :

$$\begin{cases} x_1 \quad\quad -5x_3 = 1 \\ \quad\quad x_2 + x_3 = 4 \\ \quad\quad\quad\quad 0 = 0 \end{cases}$$

x_1 and x_2 correspond to pivot columns. basic variables

The other variable x_3 is called a free variable.

If the system is consistent. we can express basis variables in terms of free variables:

$$\begin{cases} x_1 = 5x_3 + 1 \\ x_2 = 4 - x_3 \end{cases}$$

$\boxed{x_3 \text{ free}}$

Each choice (which is free) of x_3 determines a solution of the system.

Less likely to make mistakes using REF than EF in hand computations

But EF is enough to answer the existence and uniqueness questions.

Thm A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column.

If a linear system is consistent then the solution set contains

either (i) a unique sol. where $\nexists$ free var.

or (ii) infinitely many sol. when $\exists$ at least one free variable

Examp
$$\begin{cases} \quad\quad 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

Sol Augmented matrix

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

row reduced to pivots

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

- Existence? Consistent (There is no eqn. such as $0 = 1$)

- Uniqueness?

  There are free variables : x_3, x_4.

  So, there are infinitely many solutions.

REF (Backward phase)

$$\xrightarrow[\text{row 1} - 6\times\text{row 3}]{\text{row 2} - 2\times\text{row 3}}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & 9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{row 2} \div 2} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & 9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{row 1} + 9\times\text{row 2}} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & 54 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$\xrightarrow{\text{row1} \div 3}$ $\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & 18 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

Corresponding system:

$$\begin{cases} \boxed{x_1} \quad -2x_3 + 3x_4 = 18 \\ \boxed{x_2} \quad -2x_3 + 2x_4 = -7 \\ \boxed{x_5} = 4 \end{cases}$$

basic var.

$$\begin{cases} x_1 = 18 + 2x_3 - 3x_4 \\ x_2 = -7 + 2x_3 - 2x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \\ x_5 = 4 \end{cases}$$

$\boxed{4}$

Summary Using row reduction to solve a linear system.

1. Augmented matrix
2. EF. Consistent?
   If NOT, Stop; otherwise, go to the next step.
3. REF
4. Corresponding system of equations
5. Express basic variables in terms of free variables.