Systems of linear equations

A linear equation in variables $x_1, \dots, x_n$ is an equation of the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where b and coefficients $a_1, \dots, a_n$ are real or complex numbers, usually known in advance.

Notation $\mathbb{R}$: set of real #'s (numbers)

$\mathbb{C}$: set of complex #'s

Examples $3x_1 + 5x_2 = 3$

$\sqrt{2}\, x_1 + \frac{1}{3} x_2 + 4 = 4x_3$ (It can be rearranged into $\sqrt{2}\, x_1 + \frac{1}{3} x_2 - 4x_3 = -4$)

$3x + 5y = 3$ (linear equation in x and y)

Non-examples

$4x_1 + 5x_2 = x_1 x_2$

$4\sqrt{x_1} + 5x_2 = 6$ are NOT linear equations.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables — say $x_1, \dots, x_n$.

Exmp

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 11 \\ x_1 \qquad\quad - x_3 = -2 \end{cases}$$

A solution of the system is a (ordered) list of numbers $(s_1, \dots, s_n)$ satisfying the equations.

For example, $(1, 2, 3)$ is a solution of the system in the example, because

$$\begin{cases} 2 \times 1 + 3 \times 2 + 3 = 11 \\ 1 \qquad\qquad - 3 = -2 \end{cases}$$

$(1, 2, 3)$ are substituted in (x_1, x_2, x_3).

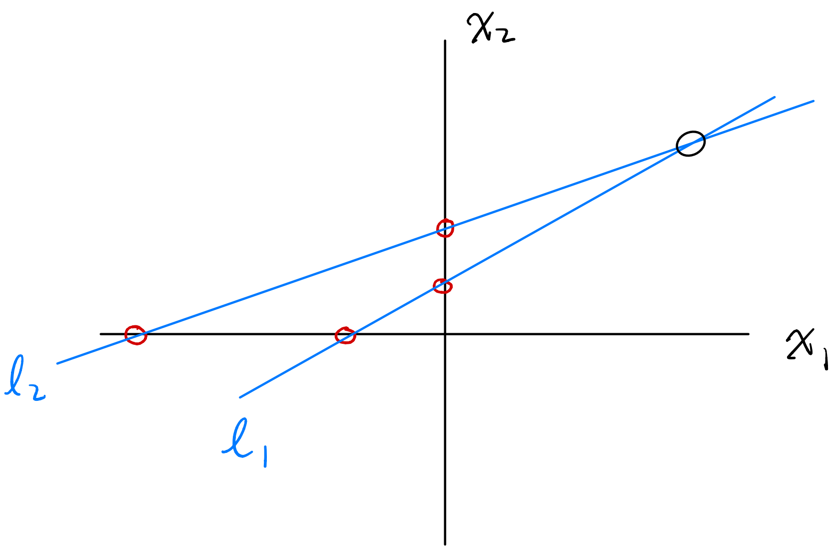
Solution set: set of all possible solutions.

Two linear systems are equivalent if they have the same solution set.

Examp

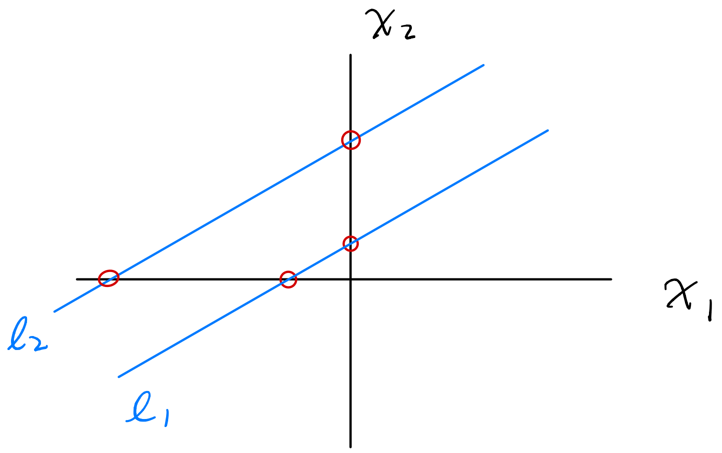
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

We can find the solutions using geometry.



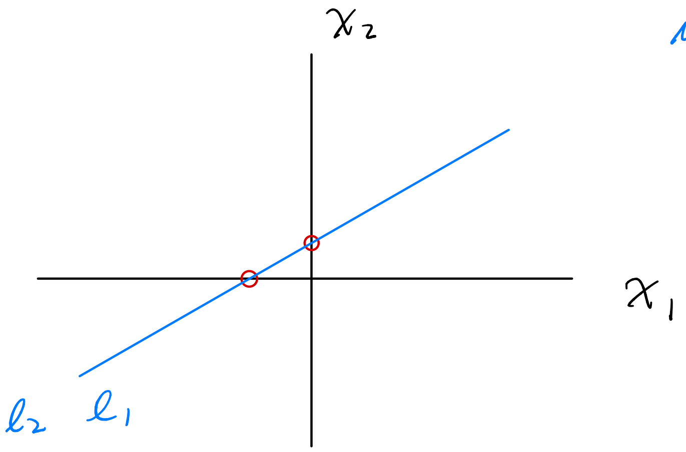
One solution.

Exmp
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases}$$



no solutions

Exmp
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$



infinitely many solutions

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions

inconsistent

$\}$ consistent.

Matrix notation

Record the essential information of a linear system in a rectangular array called a matrix.

Exmp.
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\;\; 2x_2 - 8x_3 = 8 \\ 5x_1 \qquad\;\; - 5x_3 = 10 \end{cases} \qquad (*)$$

Aligning coeff. of each var. in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the coeff matrix of the system, and the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

is called the augmented matrix of the system.

Size: $m \times n$ matrix is a rectangular array of #'s w/ m rows and n columns.

Solving a linear system

We will compare it w/ operations on matrices.

Strategy Replace one system w/ an equivalent system that is easier to solve.

Basic operations

1. Replace one eq. by the sum of itself and a multiple of another
2. Interchange two equations
3. Multiple an eqn. by a nonzero constant

They don't change the solution set.

Exmp Solve (*)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & ① \\ \qquad 2x_2 - 8x_3 = 8 & ② \\ 5x_1 \qquad\quad - 5x_3 = 10 & ③ \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Reduce the # of var. in eqn. ③ keeping others unchanged.

③ − 5 × ① :

$$\begin{array}{rrrr} & -5x_1 + 10x_2 - 5x_3 &= 0 \\ + & 5x_1 \qquad\quad - 5x_3 &= 10 \\ \hline & 10x_2 - 10x_3 &= 10 \end{array}$$

Replace the third eqn. by the new one, getting an equivalent system.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & ① \\ 2x_2 - 8x_3 = 8 & ② \\ 10x_2 - 10x_3 = 10 & ③ \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Simplify coordinates : ② ÷ 2 , ③ ÷ 10

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & ① \\ x_2 - 4x_3 = 4 & ② \\ x_2 - x_3 = 1 & ③ \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Reduce the # of variables in ③ :

③ − ② :

$$\begin{array}{rcr} -x_2 + 4x_3 & = & -4 \\ + \quad x_2 - x_3 & = & 1 \\ \hline 3x_3 & = & -3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & ① \\ x_2 - 4x_3 = 4 & ② \\ 3x_3 = -3 & ③ \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

We could clear out x_2 from ① now.

But it's more efficient to use ③ first.

Simplify coordinates : ③ ÷ 3

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & ① \\ x_2 - 4x_3 = 4 & ② \\ x_3 = -1 & ③ \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ x_2 - 4x_3 = 4 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

triangular form]
(more precise later)

(2) + 4 × (3) , (1) − (3) :

$$\begin{cases} x_1 - 2x_2 = 1 & (1) \\ x_2 = 0 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(1) + 2 × (2) : $x_1 = 1$

$$\begin{cases} x_1 = 1 & (1) \\ x_2 = 0 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Check $1 \times 1 - 2 \times 0 + 1 \times (-1) = 0$

$2 \times 0 - 8 \times (-1) = 8$

$5 \times 1 - 5 \times (-1) = 10$ ✓

Operations on the linear system $\longleftrightarrow$

operations on the augmented matrix

Elementary row operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.

2. (Interchange) Interchange two rows.

3. (Scaling) Multiply a row by a nonzero constant.

Remarks

- The operations can be applied to any matrix.
- They are reversible.

Definition Two matrices are row equivalent if there exists a sequence of elementary row operations that transforms one matrix into the other.

If the augmented matrices of two linear systems are row equivalent, then the two systems are equivalent.
(They have the same solution set)

Important to perform row operations accurately!!!

Existence & uniqueness

Two fundamental questions

1. Is the system consistent; that is, is there at least one solution?
2. If consistent, is the solution unique?

Exmp System (*) consistent. solution unique.

Exmp Consistent?

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{cases}$$

Solution (we use the augmented matrix.)

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} \xrightarrow{① \leftrightarrow ②}$$

to obtain x_1 in the first eqn.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix} \xrightarrow{③ - 2\times ①}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} \xrightarrow{③ + 2\times ②} \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

The last equation: $0 = 15$, absurd.

This means there is no solution to the system.

Inconsistent

$\boxed{\text{Q}}$