§ 6.3 Orthogonal projections (in higher dim)

Outline
- orthogonal complement
- orth. decomposition thm
- best approximation thm
- proj w/ orthonormal basis.

Similar to projecting a line, we can also project onto a larger subspace.

Thm 8 (The orthogonal decomposition thm)

$W \subseteq \mathbb{R}^n$ subspace

Then each $y \in \mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z$$

where $\hat{y} \in W$ and $z \in W^\perp$

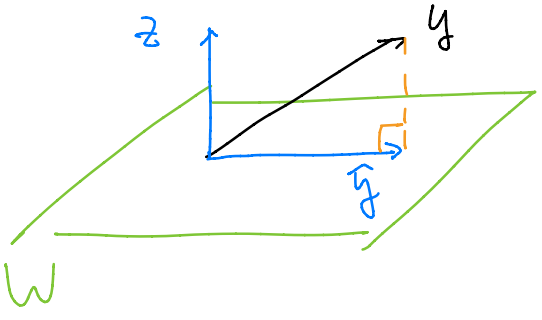
In fact, if $\{u_1, \dots, u_p\}$ is an orthogonal basis for W. then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

and $z = y - \hat{y}$.

$\hat{y}$ is the orthogonal proj of $\hat{y}$ onto W.

and denoted as $\text{proj}_W y$



Exmp 2 Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{u_1, u_2\}$ is an orthogonal basis for $W = \text{Span}\{u_1, u_2\}$

Write $y = \hat{y} + z$ s.t. $\hat{y} \in W$, $z \in W^\perp$

Sol $\hat{y} = \dfrac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \dfrac{y \cdot u_2}{u_2 \cdot u_2} u_2$

$y \cdot u_1 = 2 \times 1 + 5 \times 2 + (-1) \times 3 = 9$

$y \cdot u_2 = -2 \times 1 + 1 \times 2 + 1 \times 3 = 3$

$u_1 \cdot u_1 = 2^2 + 5^2 + (-1)^2 = 30$

$u_2 \cdot u_2 = (-2)^2 + 1^2 + 1^2 = 6$

Thus, $y = \dfrac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \dfrac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{3}{5} - 1 \\ \frac{3}{2} - \frac{1}{2} \\ -\frac{3}{10} + \frac{3}{6} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ 1 \\ \frac{1}{5} \end{bmatrix}$$

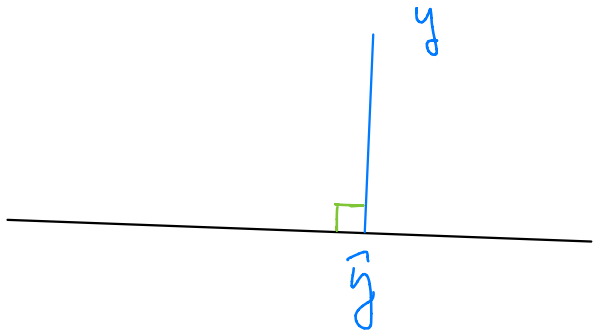
and $z = y - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 1 \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 7/5 \\ 1 \\ 14/5 \end{bmatrix}$

Check $z \in W^\perp$ ($z \perp u_1$, $z \perp u_2$) ☐

Geometric interpretation

Properties of orthogonal projections

We have seen a special version of the following theorem in plane geometry.



$\hat{y}$ is the point on L closest to y.

Thm 9 (The best approximation thm)

$W \subseteq \mathbb{R}^n$ subspace, $y \in \mathbb{R}^n$, $\hat{y} = \text{proj}_W y$.

Then $\hat{y}$ is the closest pt in W to y, namely

$$\|y - \hat{y}\| < \|y - v\|$$

for any $v \in W$ s.t. $v \neq \hat{y}$

Exmp In Exmp 2, the closest pt in W to y is

$$\hat{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1 \\ 7/5 \end{bmatrix}$$

The distance from y to W is

$$\|y - \hat{y}\| = \|z\| = \left\| \begin{bmatrix} 3/5 \\ 1 \\ 14/5 \end{bmatrix} \right\| = \sqrt{\left(\frac{3}{5}\right)^2 + 1^2 + \left(\frac{14}{5}\right)^2}$$

$$= \frac{\sqrt{9+1+256}}{5} = \frac{\sqrt{266}}{5}$$

□

If we have an orthonormal basis, the best approximation theorem simplifies.

Thm 10 $W \subseteq \mathbb{R}^n$ subspace

$\{u_1, \dots, u_p\}$ is an orthonormal basis for W, then

$\forall y \in \mathbb{R}^n$

$$\text{proj}_W y = (y \cdot u_1)u_1 + (y \cdot u_2)\cdot u_2 + \cdots + (y \cdot u_p)u_p$$

If $U = [u_1\ u_2\ \cdots\ u_p]$ ($n \times p$ matrix), then $\forall y \in \mathbb{R}^n$

$$\text{proj}_W y = UU^T y$$

Pf Notice that $u_i \cdot u_i = 1$ for $i = 1, \dots, p$

So the first statement follows from Thm 9.

On the other hand

$$UU^T y = U \begin{bmatrix} u_1^T \\ \vdots \\ u_p^T \end{bmatrix} y$$

$$= U \begin{bmatrix} u_1^T y \\ \vdots \\ u_p^T y \end{bmatrix} = U \begin{bmatrix} u_1 \cdot y \\ \vdots \\ u_p \cdot y \end{bmatrix}$$

$$= (u_1 \cdot y)u_1 + \cdots + (u_p \cdot y)u_p$$

$\square$