§6.2 Orthogonal sets

Recall basis ↔ coordinate system

Will talk about
orthogonal basis ↔ orthogonal coord. sys

Def $\{\vec{u}_1, \cdots, \vec{u}_p\} \subset \mathbb{R}^n$ is an orthogonal set if
$\forall\, i \neq j,\ \vec{u}_i \cdot \vec{u}_j = 0$.

Exmp 1 Show that $\left\{ \vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} \right.$

is an orthogonal set. (They are mutually orthogonal)

Sol $\vec{u}_1 \cdot \vec{u}_2 = 3 \times (-1) + 1 \times 2 + 1 \times 1 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = 3 \times (-\frac{1}{2}) + 1 \times (-2) + 1 \times \frac{7}{2} = 0$

$\vec{u}_2 \cdot \vec{u}_3 = (-1) \times (-\frac{1}{2}) + 2 \times (-2) + 1 \times \frac{7}{2} = 0$ ☐

Orthogonality & linear independence

Thm 4 If $S = \{\vec{u}_1, \cdots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for Span S.

Pf Suppose $c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = \vec{0}$

take their dot products w/ $\vec{u}_1$: $(c_1\vec{u}_1 + \cdots + c_p\vec{u}_p) \cdot \vec{u}_1 = \vec{0} \cdot \vec{u}_1 = 0$

On the other hand, $LHS = c_1\vec{u}_1 \cdot \vec{u}_1 + c_2\vec{u}_2 \cdot \vec{u}_1 + \cdots + c_p\vec{u}_p \cdot \vec{u}_1$

$= c_1\vec{u}_1 \cdot \vec{u}_1 + 0 + \cdots + 0$

$= c_1\vec{u}_1 \cdot \vec{u}_1$

Since $\vec{u}_1 \neq \vec{0}$, $\vec{u}_1 \cdot \vec{u}_1 \neq 0$. Therefore, $c_1 = 0$.

Similarly $c_2 = c_3 = \cdots = c_p = 0$

Hence, S is linearly independent, and S is a basis for Span S ☐

HW
6.1 14, 32, 37
6.2 7, 13

Orthogonal basis

Def An orthogonal basis for a vector space W (w/ an inner product) is a basis for W that is also an orthogonal set

Useful in writting the linear combination.

Thm 5 $\{\vec{u}_1, \dots, \vec{u}_p\}$ orthogonal basis for a subspace $W \subseteq \mathbb{R}^n$
For each $\vec{y} \in W$, the weights in the linear comb use the

(*) $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + \boxed{c_p\vec{u}_p}$ "projection onto $\vec{u}_j$" inner product on $\mathbb{R}^n$

are given by $c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$, $j = 1, \dots, p$

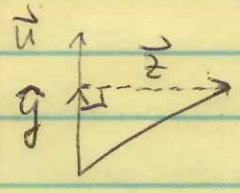
Pf Take the dot product of (*) w/ $\vec{u}_j$

$$\vec{y}\cdot\vec{u}_j = (c_1\vec{u}_1 + \cdots + c_p\vec{u}_p)\cdot\vec{u}_j$$
$$= c_j\vec{u}_j\cdot\vec{u}_j$$

1/2

An orthogonal projection

It will lead to a geometric interpretation of Thm 5.



Motivating example : decomposing a force into component forces.

Want to decompose $\boxed{\vec{y} = \hat{y} + \vec{z}}$
where $\hat{y}$ is a multiple of $\vec{u}$, and $\vec{z}$ is orthogonal to $\vec{u}$.
Look for such $\hat{y}$ and $\vec{z}$. Suppose $\hat{y} = \alpha\vec{u}$ $\alpha \in \mathbb{R}$

$$\vec{y}\cdot\vec{u} = \hat{y}\cdot\vec{u} + \vec{z}\cdot\vec{u} = \hat{y}\cdot\vec{u} = \alpha\vec{u}\cdot\vec{u}$$

Thus, $\alpha = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ and $\boxed{\hat{y} = \alpha\vec{u} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}}$ orthogonal projection of $\vec{y}$ onto $\vec{u}$.

$$\vec{z} = \vec{y} - \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

6.2

HW: 7, 15, 34

Exmp 3 $\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Find the projection of $\vec{y}$ onto $\vec{u}$. Write $\vec{y} = \hat{y} + \vec{z}$ s.t. $\hat{y} \in \text{Span}\{\vec{u}\}$ $\vec{z} \perp \vec{u}$.

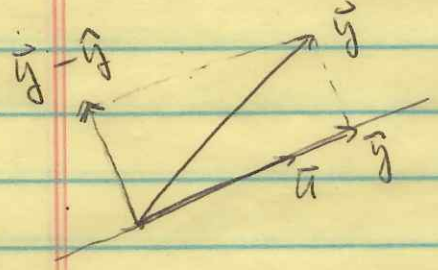
Sol projection of $\vec{y}$ onto $\vec{u}$: $\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u}$

$$= \frac{7 \times 4 + 6 \times 2}{4 \times 4 + 2 \times 2} \cdot \vec{u} = \frac{40}{20} \vec{u} = 2\vec{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Thus, $\vec{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Exmp 4 Notation as in Exmp 3. Find the distance from $\vec{y}$ to L, the line spanned by $\vec{u}$.



Sol It's given by the length of $\vec{y} - \hat{y}$:

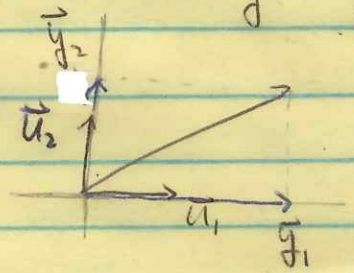
$$\|\vec{y} - \hat{y}\| = \left\| \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$

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A geometric interpretation of Thm 5

Suppose $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ where $\vec{u}_1$ & $\vec{u}_2$ are orthogonal. $\forall \vec{y} \in W$ $\vec{y}$ can be written as

$$\vec{y} = \underbrace{\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1}_{\vec{y}_1} + \underbrace{\frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2}_{\vec{y}_2}.$$



Sometimes, we want a normalized orthogonal set.

Orthonormal set

Def A set $\{\vec{u}_1, \cdots, \vec{u}_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors.

An orthonormal basis for a vector space is a basis which is also an orthonormal set.

Exmp. Standard basis of $\mathbb{R}^n$

Exmp 5. (See the textbook)

Orthogonal matrices

Matrices whose columns form an orthonormal set are important. They correspond to some linear transformations like rotations and reflections.

Thm 6 An $m \times n$ matrix has orthonormal columns iff $U^TU = I$

Pf We consider a simple case: $U = [\vec{u}_1 \ \vec{u}_2]$ $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^m$

Then $U^TU = [\vec{u}_1 \ \vec{u}_2]^T [\vec{u}_1 \ \vec{u}_2]$

$$= \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \end{bmatrix} [\vec{u}_1 \ \vec{u}_2] = \begin{bmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 \\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 \end{bmatrix}$$

Thus, $U^TU = I$ iff $\{\vec{u}_1, \vec{u}_2\}$ is orthonormal. □

The calculations above lead to

Thm 7 U = $m \times n$ matrix w/ orthonormal columns

$\vec{x}, \vec{y} \in \mathbb{R}^n$ Then

a) $\|U\vec{x}\| = \|\vec{x}\|$ (length)

b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$ (inner product)

c) $(U\vec{x}) \cdot (U\vec{y}) = 0$ iff $\vec{x} \cdot \vec{y} = 0$ (orthogonality)

The lin. trans $\vec{x} \mapsto U\vec{x}$ preserves lengths and orthogonality.

Orthogonal Complements

$W \subset \mathbb{R}^n$ subspace $\quad \vec{z} \in \mathbb{R}^n$

Def If $\vec{z}$ is orthogonal to every vector in W, then $\vec{z}$ is orthogonal to W. ($\vec{z} \perp W$)

The **orthogonal complement** of W is the set of vectors orthogonal to W in $\mathbb{R}^n$, denoted as $W^\perp$ ("W perp(endicular)").

See Figure 7 (P354)

Fact 1. $\vec{x} \in W^\perp$ iff $\vec{x}$ is orthogonal to a spanning set of W.

2. $W^\perp$ is a subspace of $\mathbb{R}^n$. Closed under addition and scalar multiplication.

Thm 3 A: $m \times n$ matrix.

$$(\text{Row}A)^\perp = \text{Nul}A, \quad (\text{Col}A)^\perp = \text{Nul}A^T$$

Pf Let $A = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}$ $\quad \vec{a}_1, \cdots, \vec{a}_m \in \mathbb{R}^n$ ($n \times 1$ matrices / vectors)

$$A\vec{x} = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1^T\vec{x} \\ \vdots \\ \vec{a}_m^T\vec{x} \end{bmatrix}$$

$$A\vec{x} = \vec{0} \Leftrightarrow \vec{a}_1^T\vec{x} = \cdots = \vec{a}_m^T\vec{x} = \vec{0} \Leftrightarrow \vec{x} \perp \vec{a}_1, \cdots \vec{x} \perp \vec{a}_m$$

$$\Leftrightarrow \vec{x} \perp \text{Row}A.$$

We can then easily deduce $(\text{Col}A)^\perp = \text{Nul}A^T$ by transposing A. ∎