§4.3 Linearly independent sets; bases

V vector space (does not have to be $\mathbb{R}^n$)

Def $\{\vec{v}_1, \cdots, \vec{v}_p\} \subseteq V$ is *linearly independent* if the vector equation

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0} \qquad (1)$$

has only the trivial solution, $c_1 = c_2 = \cdots c_p = 0$.
Otherwise, the set is *linearly dependent*, and (1) is called a *linear dependence relation* among $\vec{v}_1, \cdots, \vec{v}_p$.

Rmk For general vector spaces we may not use matrices to study (1), as what we did for $\mathbb{R}^n$.

Thm 4 An indexed set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ of 2 or more vectors, w/ $\vec{v}_1 \neq \vec{0}$, is linearly dependent iff some $\vec{v}_j$ ($j > 1$) is a linear comb. of $\vec{v}_1, \cdots, \vec{v}_{j-1}$.

Exmp $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = 4 - t$ (in $\mathbb{P}_1$)
Then $\{p_1, p_2, p_3\}$ is linear dependent since $p_3 = 4p_1 - p_2$

Def (Basis) V vector space. $B \subset V$ a set of vectors
B is a basis for V if
(i) B is lin. ind.
(ii) B spans V

Exmp A: inv $n \times n$ matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$
Then by the Invertible Matrix Thm.
$\{\vec{a}_1, \cdots, \vec{a}_n\}$ forms a basis for $\mathbb{R}^n$.
- lin. ind.
- They span $\mathbb{R}^n$

Exmp (Standard basis for $\mathbb{R}^n$)

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Exmp Determine if $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

form a basis for $\mathbb{R}^3$

NO, because they are not linearly independent:

$\vec{v}_1 + \vec{v}_3 = 2\vec{v}_2$

(We can also see this via the determinant $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

$= 0$

The spanning set thm

discard redundant vectors

Exmp Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

and $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Note that $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$ and show that $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

Then find a basis for H.

Sol $\text{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

On the other hand, any lin. comb.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(5\vec{v}_1 + 3\vec{v}_2)$$

$$= (c_1 + 5c_3)\vec{v}_1 + (c_2 + 3c_3)\vec{v}_2 \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$$

Thus, $\text{Span}\{\vec{v}_1, \vec{v}_2\} \supseteq \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

v_1, v_2 are lin. ind. Thus, they form a basis for H.

We don't need $\vec{v}_3$.

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Thm 5 $S = \{\vec{v}_1, \cdots, \vec{v}_p\} \subset V$. $H = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$.

a. If $\vec{v}_k$ is a linear comb. of other vectors in S. then $S \setminus \{\vec{v}_k\}$ still spans H.

b. If $H \neq \{\vec{0}\}$. some subset of S is a basis for H

Bases for Nul A, Col A. Row A.

We have seen how to find bases for Nul A and Col A in the last lecture.

Thm 7 If two matrices A & B are row equivalent. then Row A = Row B

If B is in echelon form, the nonzero rows of B form a basis for Row B and Row A.