§4.2 Null / Column / Row spaces & linear transformations

Null spaces

A $m \times n$ matrix

Note this

Def (Null space) $\text{Nul } A = \{ \vec{x} \mid \vec{x} \in \mathbb{R}^n,\ A\vec{x} = 0 \}$

"implicitly defined"

If we want to determine whether a vector $\vec{u}$ is in Nul A we just compute $A\vec{u}$.

Thm 2 Nul A is a subspace of $\mathbb{R}^n$

An explicit description of Nul A

i.e. solving the equation $A\vec{x} = 0$.

Example 3 Find a spanning set for Nul A: | Write the sol's in parametric forms.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Sol Find the general sol. of $A\vec{x} = 0$ in terms of free var's.

Aug. mat.: $[A\ \mathbf{0}]$

Reduced echelon form: (p.c, p.c)

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear system:

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{cases}$$

Free variables: x_2, x_4, x_5

General sol:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

($\vec{u}$, $\vec{v}$, $\vec{w}$)

Actually, they form a basis.

↳ Spanning set: $\vec{u}, \vec{v}, \vec{w}$

(1)

Column space A $m \times n$ matrix

$A = [\vec{a}_1 \cdots \vec{a}_n]$ $\quad \vec{a}_i \in \mathbb{R}^m$ $\quad i = 1, \cdots, n$.

Def $\operatorname{Col} A = \operatorname{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$

$= \{\vec{b} \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$

$\vec{b} \in \mathbb{R}^m$

Recall In order to find a basis for $\operatorname{Col} A$,

look for the pivot columns of A. The original A.

NOT its REF.

Exmp.

In the previous exmp.

$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ form a basis for $\operatorname{Col} A$

Row space

$A = \begin{bmatrix} - \vec{b}_1 - \\ \vdots \\ - \vec{b}_m - \end{bmatrix}$ $\vec{b}_i$ are vectors written horizontally.

Then $\operatorname{Row} A = \operatorname{Span}\{\vec{b}_1, \cdots, \vec{b}_m\}$

Exmp 6 - 8.

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Linear transformations

Def $T: V \to W$, a map (a rule that assigns to each vector $\vec{x} \in V$ a unique vector $T(\vec{x}) \in W$.) between vector spaces, is a linear transformation if

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall\ \vec{u}, \vec{v} \in V.$

(ii) $T(c\vec{u}) = cT(\vec{u}) \quad \forall\ \vec{u} \in V,\ c \in \mathbb{R}$ (scalar)

kernel (or null space) of $T = \ker T = \{ \vec{u} \in V \mid T(\vec{u}) = \vec{0} \}$

range of $T = T(V) = \{ T(\vec{x}) \mid \vec{x} \in V \}$

Thm $\ker T$ is a subspace of V.

$T(V)$ is a subspace of W.

Exmp (Derivative)

$\mathbb{P}_2 = \{$ poly. in t of degree at most $2\}$

$D: \mathbb{P}_2 \to \mathbb{P}_2$

$a + bt + ct^2 \longmapsto b + 2ct$

- D is a linear transformation

Given two polynomials: $p(t) = a_1 + b_1 t + c_1 t^2$

$q(t) = a_2 + b_2 t + c_2 t^2$

$D(p(t)+q(t)) = D(a_1 + a_2 + (b_1+b_2)t + (c_1+c_2)t^2)$

$= b_1 + b_2 + 2(c_1+c_2)t$

$D(p(t)) + D(q(t)) = (b_1 + 2c_1 t) + (b_2 + 2c_2 t) = D(p(t)+q(t))$

Similarly, $D(\lambda p(t)) = \lambda D(p(t))$ for any polynomial $p(t)$ and scalar λ.

- kernel $\ker D = \{$ constant polynomials$\}$
- range $D(\mathbb{P}_2) = \{$ poly. of deg at most $1\}$