- Vector spaces
- subspaces
- Span.

§4.1 Vector spaces and subspaces

Many mathematical systems have similar structure as $\mathbb{R}^n$.
Here is the abstract definition.

Def A vector space is a nonempty set V on which addition and scalar multiplication are defined such that. $\forall \vec{u}, \vec{v}, \vec{w} \in V$, $c, d \in \mathbb{R}$

1. $\vec{u} + \vec{v} \in V$ (addition)
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity)
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity)
4. $\exists\ \vec{0} \in V$ s.t. $\vec{u} + \vec{0} = \vec{u}$. (zero vector)
5. $\exists\ -\vec{u} \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$ ($-\vec{u}$ is the negative of $\vec{u}$)
   ($-\vec{u}$ is a vector)
6. $c\vec{u} \in V$ (scalar mult.)
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ } distributivity.
8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$.

Basic facts

- $\vec{0}$ is unique
- $-\vec{u}$ is unique for each $\vec{u}$.
- $0 \cdot \vec{u} = \vec{0}$
- $c \cdot \vec{0} = \vec{0}$
- $-\vec{u} = (-1)\vec{u}$

Exmp $\mathbb{R}^n$

$\mathbb{S}$ = set of

Exmp 3 (signals) doubly infinite sequences of numbers:

$(\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2 \cdots)$

$(\cdots, z_{-2}, z_{-1}, z_0, z_1, z_2, \cdots)$ another element.

Their sum = $(\cdots, y_{-2}+z_{-2}, y_{-1}+z_{-1}, y_0+z_0, y_1+z_1, y_2+z_2, \cdots)$

scalar mult. $c \cdot (\cdots, y_{-1}, y_0, y_1, \cdots) = (\cdots, cy_{-1}, cy_0, cy_1, \cdots)$

$\mathbb{S}$ is a vector space.

. For example, zero vec. = $(\cdots, 0, 0, 0, \cdots)$.

Exmp (polynomials) $\mathbb{P}$ = set of all polynomials in the variable t w/ real coefficients

A polynomial in $\mathbb{P}$ can be written as

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots$$

Another polynomial $q(t) = b_0 + b_1 t + b_2 t^2 + \cdots$

addition: $(p+q)(t) := p(t) + q(t) = (a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2 + \cdots$

scalar mult: $c \in \mathbb{R}$. $(cp)(t) = c(p(t)) = ca_0 + ca_1 t + ca_2 t^2 + \cdots$

$\mathbb{P}$ is a vector space.

Exmp (Function space) V = set of real valued functions on a set D

addition: $f, g \in V$, $(f+g)(t) := f(t) + g(t)$

scalar mult. $c \in \mathbb{R}$, $f \in V$, $(cf)(t) := c\,f(t)$.

V is a vector space.

Ex: Check that they are v.s.

Subspaces

"smaller vector spaces in a larger vector space."

We have seen exmp's : Col spaces, Null spaces.

Def A subspace of a v.s. V is a subset H of V s.t.

a) $\vec{0}$ of V is in H

b) H is closed under addition: $\forall\ \vec{u}, \vec{v} \in H,\ \vec{u}+\vec{v} \in H$

c) H —— scalar mult: $\forall\ c \in \mathbb{R},\ \vec{u} \in H,\ c\vec{u} \in H$

Rmk Other axioms are automatically true for H.

Exmp $\{\vec{0}\} \subset V$ is a subspace, the trivial/zero subspace.

Exmp $\{$ polynomials of degree at most $n\} \subset \mathbb{P}$ is a subspace
(real)

Span

V v.s., $\vec{v}_1, \cdots, \vec{v}_p \in V$.

$\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$ = set of all linear combs of $\vec{v}_1, \cdots, \vec{v}_p$.

$= \{ c_1\vec{v}_1 + \cdots + c_p\vec{v}_p \mid c_1, \cdots, c_p \in \mathbb{R}\}$

Thm $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$ is a subspace of V.

Review §2.8