§3.3 Cramer's rule, linear transformations

Plan
- Cramer's Rule
- A^{-1}
- Linear trans.

$A\vec{x} = \vec{b}$ $\quad A$: $n \times n$ matrix

$A_i(\vec{b})$: $n \times n$ matrix obtained from A by replacing column i by the vector $\vec{b}$.

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \underset{\text{col. } i}{\vec{b}} \cdots \vec{a}_n]$$

Write $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Thm 7 (Cramer's Rule) $A\vec{x} = \vec{b} \in \mathbb{R}^n$. A invertible. Then the unique solution $\vec{x}$ has entries

$$x_i = \frac{\det(A_i(\vec{b}))}{\det A} \qquad i = 1, 2, \ldots, n.$$

Exmp 1 $\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$

Sol. Coeff matrix $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, $\det A = 2 \neq 0$ A inv.

$$A_1(\vec{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \qquad A_2(\vec{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det A_1(\vec{b})}{2} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det A_2(\vec{b})}{2} = \frac{24 + 30}{2} = 27$$

13.3

A formula for A^{-1}

(Using Cramer's Rule) Suppose A $n \times n$ inv. matrix.

Write $A^{-1} = [\vec{u}_1 \cdots \vec{u}_n]$

$$AA^{-1} = A[\vec{u}_1 \cdots \vec{u}_n] = I_n = [\vec{e}_1 \cdots \vec{e}_n]$$

Thus, for $i = 1, \ldots, n$, $A\vec{u}_i = \vec{e}_i$ (need to solve)

Cramer's Rule $\Rightarrow$ the j-th entry of $\vec{u}_i$ is $\dfrac{|A_j(\vec{e}_i)|}{|A|}$

$$A_j(\vec{e}_i) = \begin{bmatrix} & \vdots & \\ \cdots & 0 & \cdots \\ & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \\ \cdots & 0 & \cdots \\ & \vdots & \end{bmatrix} \leftarrow \text{row } i$$

$\uparrow$ col. j

Cofactor expansion of $A_j(\vec{e}_i)$ down the j-th col.:

$$|A_j(\vec{e}_i)| = (-1)^{i+j} A_{ij} = C_{ij}$$

Thus, the (j,i)-entry of A^{-1} is $\dfrac{C_{ij}}{|A|}$.

That is,

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & \vdots \\ \vdots & \vdots & & \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

adjugate of A denoted as $\operatorname{adj} A$.

Linear transformations $n = 2$ or 3

Thm 10 $T : \mathbb{R}^n \to \mathbb{R}^n$ linear trans w/ standard matrix A. $S \subset \mathbb{R}^n$ parallelogram / parallelepiped.

$T(S)$ = set of images of points in S

Then,

area / volume of $T(S)$ = $|\det A| \cdot$ (area / vol. of S)

↑ absolute value of $\det A$

Rmk It can be applied to other regions ~~w/~~ finite area / vol. as well.