Outline - matrix representation of a linear transformation
- lin. trans. of $\mathbb{R}^2$
- Existence and uniqueness questions.

☆ Important and difficult.

§1.9 The matrix of a linear transformation

We have seen that $L_A : \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$ is a lin. trans

Actually, all lin. trans $\mathbb{R}^n \to \mathbb{R}^m$ is of this form.

Goal $A = ?$

! - A linear transformation is determined by the images of a few vectors

$$I_n = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

$\vec{e}_1 \; \vec{e}_2 \; \cdots \; \vec{e}_n$ ← These are the vectors

Exmp. 1 $T : \mathbb{R}^2 \to \mathbb{R}^3$ lin. trans. s.t.

$$T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

arbitrary

$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ?$$

T linear

$$\Rightarrow T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = T(x_1\vec{e}_1) + T(x_2\vec{e}_2) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$$

T is completely determined by $T(\vec{e}_1)$ & $T(\vec{e}_2)$

Thm 10 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\exists!$ (there exists a unique) matrix A s.t.

$$T(\vec{x}) = A\vec{x} \quad \forall\, x \in \mathbb{R}^n$$

(for all)

In fact $A = [\,T(\vec{e}_1) \cdots T(\vec{e}_n)\,]$

Def A is the standard matrix for the linear trans. T.

Omit. Pf (w/o uniqueness) $T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$

$$= [\,T(\vec{e}_1) \; T(\vec{e}_2) \cdots T(\vec{e}_n)\,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x} \quad \blacksquare$$

Exmp 2 Find the standard matrix for dilation $T(\vec{x}) = 3\vec{x}$ for $\vec{x} \in \mathbb{R}^2$

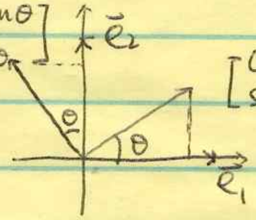
Sol We have $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(\vec{e}_1) = 3\vec{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad T(\vec{e}_2) = 3\vec{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Thus $A = [\,T(\vec{e}_1)\ \ T(\vec{e}_2)\,] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3I_2$ $\boxed{\text{☑}}$

Exmp 3 $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation around the origin through an angle φ. counter-clockwise.

$T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = T(\vec{e}_1)$



The standard matrix for T is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Some common linear trans. of $\mathbb{R}^2$

Table 1-4

Existence and uniqueness of solutions

This can be translated into questions about linear trans.

Def A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if each $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$, that is the range of T is $\mathbb{R}^m$.

related to existence

☆ solutions are the vectors mapped to $\vec{b}$

If T is onto, then $T\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

If not, then $T\vec{x} = \vec{b}$ has no solution for $\vec{b}$ outside of the range of T.

Exmp $T: \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ NOT onto.

Def T is one-to-one if each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$

related to uniqueness

If T is one-to-one, then $T\vec{x} = \vec{b}$ has at most one solution

Exmp $T: \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1$ NOT one-to-one

To see one-to-one, we only need to focus on one equation.

Thm 11 $T: \mathbb{R}^n \to \mathbb{R}^m$ lin. trans. Then

T one-to-one iff $T(\vec{x}) = \vec{0}$ has only the trivial solution.

$\Longleftrightarrow$

Why?

Omit

($\Rightarrow$) (Suppose T is one-to-one) only one vector is mapped to $\vec{0}$.

Thus $T(\vec{x}) = \vec{0}$ has only one solution, the trivial solution.

($\Leftarrow$) (Suppose $T(\vec{x}) = \vec{0}$ has only the trivial solution.)

If $T(\vec{x}_1) = T(\vec{x}_2) = \vec{b}$, then $T(\vec{x}_1 - \vec{x}_2) = T(\vec{x}_1) - T(\vec{x}_2) = \underbrace{\vec{b} - \vec{b}}_{=\vec{0}}$

Thus $\vec{x}_1 = \vec{x}_2$. T is one-to-one $\boxed{}$.

determine "onto" & "one-to-one" using the standard matrix.

Thm 12 $T: \mathbb{R}^n \to \mathbb{R}^m$ lin. trans. w/ standard matrix A.

a) T onto iff columns of A span $\mathbb{R}^m$ "enough"

b) T one-to-one iff columns of A are linearly independent. "no redundancy"

Omit

Pf a) T onto $\Leftrightarrow$ $\forall \vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a sol. say $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\hookrightarrow A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{b}$

$\Leftrightarrow$ $a_1, \ldots, a_n$ span $\mathbb{R}^m$.

b) T 1-1 $\Leftrightarrow$ $T(\vec{x}) = \vec{0}$ has only sol. $\vec{0}$.

$\Leftrightarrow$ $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0}$ has only sol. $\vec{0}$.

$\Leftrightarrow$ They are linearly indep. $\boxed{}$

Exmp $T: \mathbb{R}^2 \to \mathbb{R}^2$ lin trans w/ standard matrix

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It's the projection to the first coord.

It's neither onto or one-to-one $\boxed{}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$