§1.8 Intro to linear transformations

Def A transformation (or mapping) T from $\mathbb{R}^n$, $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.
We denote it as

$$T: \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{codomain}}$$

$T(x)$: image of x
range of T = set of all images $T(x)$

We have seen examples
Matrix transformations

A : $m \times \boxed{n}$ matrix (w/ real entries)

(left) multiplication by A. $L_A : \underbrace{\mathbb{R}^{\boxed{n}}}_{\text{domain}} \to \mathbb{R}^{m}$ ← codomain

$\vec{x} \longmapsto A\vec{x} = L_A(x)$ meaning $A\vec{x}$ is the image of $\vec{x}$.

- The range of L_A is the span of columns of A.

We have seen that
- $L_A(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = L_A(\vec{u}) + L_A(\vec{v})$
- $L_A(c\vec{u}) = A(c\vec{u}) = c\,L_A(\vec{u})$

We have the following general definition

(abstract) **Def** A transformation T is linear if
(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
(ii) $T(c\vec{u}) = c\,T(\vec{u})$

Intentionally vague about the (co)domain of T. They may not be $\mathbb{R}^n$ or $\mathbb{R}^m$.

Rmk T may no longer be in the form of matrices. For example, taking the derivative of a differentiable function.

Properties T linear transformation then

$$T(\vec{0}) = \vec{0}, \quad T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = c_1 T(\vec{v}_1) + \cdots + c_p T(\vec{v}_p).$$

$r \in \mathbb{R}$ scalar

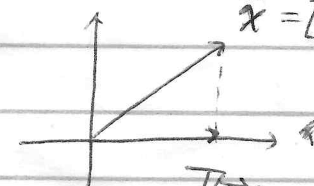
<u>Exmp</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto r\vec{x}$ is a linear transformation.

- When $0 \le r < 1$, T is a contraction
- when $r > 1$, T is a dilation.

<u>linear</u>
$$T(\vec{x}+\vec{y}) = r(\vec{x}+\vec{y}) = r\vec{x}+r\vec{y} = T(\vec{x})+T(\vec{y})$$
$$T(c\vec{x}) = r(c\vec{x}) = c(r\vec{x}) = cT(\vec{x})$$

<u>Exmp</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$
$$\vec{x} \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}$$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$T(\vec{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

Proj $\vec{x}$ to the first coord. axis.

multiplication by a matrix also <u>linear</u>.

<u>Mappings vs. equations</u>

<u>Exmp</u> $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{x} \mapsto A\vec{x}$.

a) $T(\vec{u}) = ?$ $\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$
image

b) Find an $\vec{x}$ whose image under T is $\vec{b}$.
that is $T(\vec{x}) = A\vec{x} = \vec{b}$

c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?
uniqueness of solution

d) $\vec{c} \in$ range of T?
that is $T(\vec{x}) = A\vec{x} = \vec{c}$ have a solution.
does

Read the textbook.

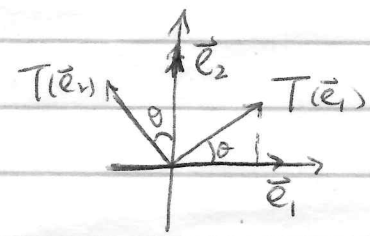
Skip.

Exmp. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{x} \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



rotation counterclockwise by angle θ.