§ 1.7 Linear independence

Given a plane, we want to set up a coordinate system.

This is closely related to the following concept.

Def An indexed set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is *linearly independent* if the vector equation

linear dependence relation

$$x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = 0$$

has only the trivial solution.

The set of vectors is *linearly dependent* if $\exists$ weights $c_1, \dots, c_p$ not all zero s.t.

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = 0.$$

Exmp $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(a) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. dependent?

(b) If possible, find a linear dep. relation.

Sol solve $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$

$\rightsquigarrow$ Aug. mat. $\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$

$\sim$ EF $\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

There is a free variable x_3. Therefore, there are nontrivial sol.

(a) Yes.

(b) $\sim$ REF $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$

x_3 can take any value. Say $x_3 = 1$. Then $x_1 = 2$, $x_2 = -1$

Thus $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

We can similarly discuss the linear (in)dependence of column vectors of a matrix.

Sets of one vector

$\{\vec{v}\}$ is linear independent iff $\vec{v} \neq \vec{0}$.
dependent iff $\vec{v} = \vec{0}$

This is because $x\vec{v} = 0$ has a nontrivial solution (x) iff $\vec{v} = \vec{0}$

Sets of two vectors

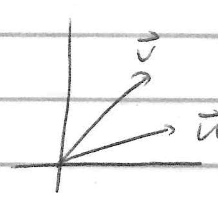
- $\{\vec{u}, \vec{v}\}$ is linearly independent iff neither of the vectors is a multiple of the other.
- ——————— dependent iff at least one of the vector is a multiple of the other.

Exmp. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly dependent.

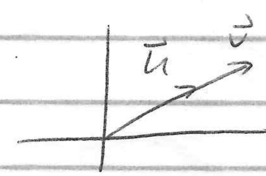
(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ —— independent

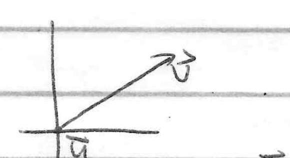
(c) $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— dependent

Geometrically

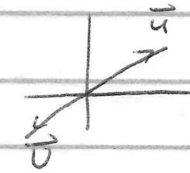


independent





dependent.



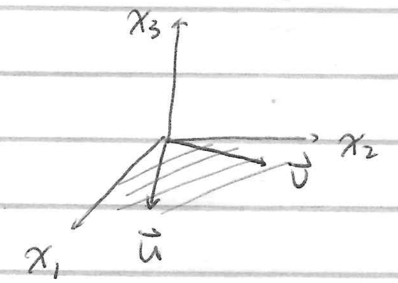
Sets of two or more vectors

Thm $S = \{\vec{v}_1, \dots, \vec{v}_p\}$, $p \geq 2$, is linearly dependent iff at least one of the vectors in S is a linear comb. of the others. (In fact, if S is linear dependent and $\vec{v}_1 \neq 0$, then some v_j $(j > 1)$ is a linear comb. of $\vec{v}_1, \dots, \vec{v}_{j-1}$

Exmp $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$ Span$\{\vec{u}, \vec{v}\} = ?$

Explain. $\vec{w} \in$ Span$\{\vec{u}, \vec{v}\}$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

Sol

- 

  Span$\{\vec{u}, \vec{v}\}$ = $x_1 x_2$-plane.

- If $\vec{w} \in$ Span$\{\vec{u}, \vec{v}\}$ then $\vec{w} = a\vec{u} + b\vec{v}$ for some $a, b \in \mathbb{R}$
  Thus, they are linear dependent.
- If they are dependent, then $\exists\ a, b, c \in \mathbb{R}$ not all zero s.t. $a\vec{u} + b\vec{v} + c\vec{w} = 0$.
  $c \neq 0$ Otherwise, $a\vec{u} + b\vec{v} = 0$ impossible.
  Thus, $\vec{w} = \frac{a}{c}\vec{u} + \frac{b}{c}\vec{v}$. Hence $\vec{w} \in$ Span$\{\vec{u}, \vec{v}\}$. $\square$

Some linear dependent examples

☆ **Thm** $\{\vec{v}_1, \dots, \vec{v}_p\} \subset \mathbb{R}^n$ lin. dependent if $p > n$.

Important for later applications

"Too many vectors"

e.g. We don't need 3 coordinates for a plane.

$$x_1 \vec{v}_1 + \cdots + x_p \vec{v}_p = \vec{0}.$$

Pf Aug mat. $[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_p \mid 0]$

There is at least one free. var.

They are dependent.

Thm If $\{v_1, \ldots, v_p\} \subset \mathbb{R}^n$ contains $\vec{0}$, then they are dependent.

See Examp. 6.