§1.5 Solution sets of linear systems

- homogeneous linear sys
- Non homogeneous systems
- structure

Homogeneous linear systems

A linear system is homogeneous if it has the form

$A\vec{x}=\vec{0}$ (as matrix eqn)

Note It always has a solution $\vec{0}$. (trivial solution)

Q Nontrivial solution?

Existence and Uniqueness Thm in §1.2 $\Rightarrow$

Fact The eqn. $A\vec{x}=\vec{0}$ has a nontrivial sol. iff it has at least one free variable.

Exmp . . . , . . Nontrivial sol.? Sol set?

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = \boxed{0} \\ -3x_1 - 2x_2 + 4x_3 = \boxed{0} \\ 6x_1 + x_2 - 8x_3 = \boxed{0} \end{cases}$$

homogeneous

Sol Aug. mat.: $[A\ 0] = \left[\begin{array}{ccc} \underbrace{\begin{matrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{matrix}}_{A} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$

$\sim$ EF echelon form $\begin{bmatrix} \boxed{3} & 5 & -4 & 0 \\ 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

ps: p.c.

x_3 is a free variable. Thus, $\exists$ nontrivial sol.

$\sim$ REF $\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{cases} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \end{cases}$

As a vector, the general sol. has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Every sol. is a multiple of $\vec{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$.
(trivial or not)

The Sol. set is a line through $\vec{0}$ in $\mathbb{R}^3$.
(& $\vec{v}$)

Exmp $\quad 2x_1 - 4x_2 - 2x_3 = 0$

Aug. mat. $[A \;\; 0] = [2 \;\; -4 \;\; -2 \;\; 0] \sim$ REF $= [\boxed{1} \;\; -2 \;\; -1 \;\; 0]$
p.c.

x_2, x_3 free variables

$x_1 = 2x_2 + x_3$

free parameters

general sol. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_3 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$ parametric form

sol. set $= \{$lin. comb. of $\vec{v}_1$ & $\vec{v}_2\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$
$=$ plane through $\vec{0}$, $\vec{v}_1$ & $\vec{v}_2$ in $\mathbb{R}^3$.

- In general, sol. set of (homogeneous) $A\vec{x} = \vec{0}$ ~~is~~ $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ for suitable vectors ~~&~~ $\vec{v}_1, \dots, \vec{v}_p$.

Solutions of non-homogeneous systems

$A\vec{x} = \vec{b}$, $\vec{b} \neq 0$ (*)

Suppose we have two distinct solutions $\vec{u}$ & $\vec{v}$, that is $A\vec{u} = \vec{b}$, $A\vec{v} = \vec{b}$

Then $A(\vec{u} - \vec{v}) = A\vec{u} - A\vec{v} = \vec{b} - \vec{b} = 0$

That is the difference $\vec{u} - \vec{v}$ of two solutions is the sol. of the corresponding homogeneous equation $A\vec{x} = 0$. (**)

Moreover, $\vec{u} = \vec{v} + (\vec{u} - \vec{v})$

$\vec{v}$: arbitrary sol. of (*); $(\vec{u} - \vec{v})$: arbitrary sol. of (**)

See Thm 6.

Exmp 3 $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & 8 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$

Sol

$$[A \ \vec{b}] = \begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & 4 \end{bmatrix} \overset{\text{row reduction}}{\sim} \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 free variable.

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \\ 0 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

Sometimes, we replace this by another letter, say t, to indicate that it's a free variable.

Check $\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -3 + 10 + 0 \\ 3 - 4 + 0 \\ -6 + 2 + 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

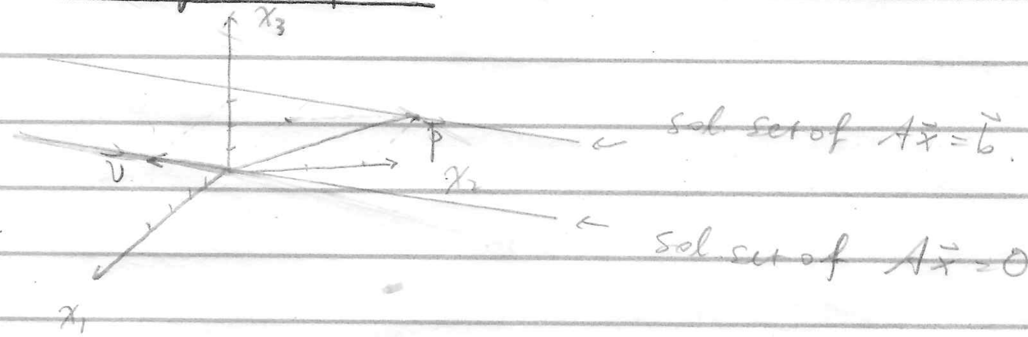
Solves $A\vec{x} = \vec{b}$.

(to be cont'd)

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 \\ -4+4 \\ 8-8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ solves $\vec{A}\vec{b} = \vec{0}$. $\boxed{4}$

Geometric descriptions



Writing a sol. set (of a consistent sys.) in parametric vector form.

1. Aug. mat $\sim$ REF
2. Express each basic var. in terms of free var's.
3. Write a typical sol. $\bar{x}$ as a vector whose entries depend on possible free var.
4. Decompose x into a linear combination of vectors using free var. as parameters.