## §1.4 The matrix equation $A\vec{x} = \vec{b}$

**Def** If $A$ is an $m \times n$ matrix, w/ col's $\vec{a}_1, \dots, \vec{a}_n$, $\vec{x} \in \mathbb{R}^n$ $= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then the product of $A$ and $\vec{x}$ is

$$A\vec{x} = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

linear combination of col. vectors.

Exmp

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

Exmp $3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3 = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$

### Linear systems as matrix equations

Exmp
$$\begin{cases} x_1 + 2x_2 - x_3 = 4 \\ \quad -5x_2 + 3x_3 = 1 \end{cases}$$
equivalent to

$$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$
equivalent to

the matrix equation
$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\text{constants}}$$

More generally, we have

Important!!! **Thm** $A$ : $m \times n$ matrix w/ col's $\vec{a}_1, \dots, \vec{a}_n$. $\vec{b} \in \mathbb{R}^m$. Then the matrix equation

$$A\vec{x} = \vec{b}$$

has the same solution set as vector eqn.

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}.$$

the same soln set as the linear sys. w/ aug. matrix

$$[\vec{a}_1, \dots, \vec{a}_m \ \vec{b}]$$

Existence of soln's

Observation $\quad A\vec{x} = \vec{b}$ has a soln iff $\vec{b}$ is a linear comb. of col. vectors of $A$.

A related but harder question:

Is the equation $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$?

Equivalent question: ~~Is $\vec{b}$ in the~~ Span{columns of $A$} $= \mathbb{R}^m$?

($A$ is an $m \times n$ matrix)

Exmp $\quad A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Is the equation $A\vec{x} = \vec{b}$ consistent for all possible $b_1, b_2, b_3$? that is $\vec{b}$

Sol. row reduce aug. matrix $[A \ \vec{b}]$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

Last column a pivot column?

Possibly. For example $b_3 = 1$, $b_1 = b_2 = 0$.

Answer: NO. 12

Rmk The three column vectors only span a plane in $\mathbb{R}^3$, not the entire $\mathbb{R}^3$.

in $\mathbb{R}^m$

(In general) Defn A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ spans $\mathbb{R}^m$ if $\text{Span}\{\vec{v}_1, \dots, v_p\} = \mathbb{R}^m$, that is every vector in $\mathbb{R}^m$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_p$.

Thm $\quad A$: $m \times n$ matrix. Then the following statements are equivalent

(a). $A\vec{x} = \vec{b}$ consistent for each $\vec{b} \in \mathbb{R}^m$

(b). Each $\vec{b} \in \mathbb{R}^m$ is a linear comb. of columns of $A$.

(c) The columns of $A$ span $\mathbb{R}^m$

(d) $A$ has a pivot position in every ROW.

They are either all true or all false.

Look at the exmp

Back to the computation of $A\vec{x}$

Exmp $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Recall $A\vec{x}$ is defined only when

# of columns in $A$ = # of rows in $\vec{x}$

In another form

$$\begin{array}{lcc} & A & \vec{x} \\ \text{type:} & m \times \underline{n} & \underline{n} \times 1 \end{array}$$

$A\vec{x}$ is defined only when these two #'s are the same.

(continued)

$$A\vec{x} = x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{2x_1 + 3x_2 + 4x_3} \\ -x_1 + 5x_2 - 3x_3 \end{bmatrix}$$

The first entry in $A\vec{x}$ is ~~by multiplying the correspon~~ a sum of products using the first row of $A$ and the entries in $\vec{x}$.

This is called the Row-vector rule for computing $A\vec{x}$.

If $A\vec{x}$ is defined, then the $i$th entry of $A\vec{x}$ is the sum of products of corresponding entries from row $i$ of $A$ and from the vector $\vec{x}$.

Exmp (see next page)

Properties of operations in $\mathbb{R}$ induces $\boxed{\text{important}}$ properties:

$A$ $m \times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$. then

LINEAR
- $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\vec{u}) = cA\vec{u}$

verify
then they
are defined.

Exmp.

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1\times 4 + 2\times 3 + (-1)\times 7 \\ 0\times 4 + (-5)\times 3 + 3\times 7 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\times 4 + (-3)\times 7 \\ 8\times 4 + 0\times 7 \\ -5\times 4 + 2\times 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\boxed{\text{identity matrix}}$ ($3\times 3$ in this instance)

$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}_{m\times n}$ (1's on the diagonal, 0's everywhere else)

$m\times m$ identity matrix